Problem 9B,1
Suppose ν is a real measure on a measurable space (X,S). Prove that the Hahn decomposition is almost unique in the sense that if A,B and $A^{'},B^{'}$ both satisfies the condition in Hahn decomposition, then

$$|\nu|(A-A^{'}) = |\nu|(A^{'}-A) = |\nu|(B-B^{'}) = |\nu|(B^{'}-B) = 0$$

Proof. We only prove that $|\nu|(A-A^{'}) = 0$. In fact, for any subset $E \subset A-A^{'}$, $\nu(E)$ should be both nonpositive and nonnegative as a subset of $B^{'}, A$. So $\nu(E) = 0$. Then the result follows. □

Problem 9B,2
Suppose ν is a positive measure and $g,h \in L^1(d\nu)$. Prove that $gd\mu \perp hd\mu$ if and only if $g(x)h(x) = 0$.

Proof. By definition, we can check that the support of measure $fd\mu$ is $\{x|f(x) \neq 0\}$. Then the result follows easiliy. □

Problem 9B,11
Suppose μ is a positive measure on a measurable space (X,S) and ν is a real measure.Show that $\nu \ll \mu$ if and only if $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.

Proof. Obviously by definition if $\nu^+ \ll \mu$ and $\nu^- \ll \mu$, then $\nu \ll \mu$. Conversely, if $\nu \ll \mu$, let $\nu = \nu^+ - \nu^-$ be its decomposition and A,B be the corresponding support set. For any set E such that $\mu(E) = 0$. Then $\mu(E \bigcap A) = \mu(E \bigcap B) = 0$. Then $\nu^+(E) = \nu(E \bigcap A) = 0$ and similarly for ν^-. This shows the result. □

Problem 9B,13
Give an example to show that Radon-Nikodym theorem can fail if the σ-finite hypothesis is eliminated.

Proof. Let $X = [0,1], S = \mathcal{B}_{[0,1]}$, and let m be the Lebesgue measure and μ be the counting measure. Then obviously $m \ll \mu$. But if there exists f such that $dm = fd\mu$, then obviouly $f \geq 0$. Take single point as the test sets and we know that f must be zero function, which is an obviously contradiction. Note we can easily check that counting measure does not satisfy the σ- finite condition. □

Problem 10A,3
Suppose V,W are Hilbert space and $g \in V, h \in W$. Define $T \in B(V,W)$ by $Tf =< f,g > h$. Find a formula for T^*.

Proof.

$$< Tf,k >=< f,g >< h,k >=< f,< k,h > g >=< f,T^*k >,$$

thus $T^*k =< k,h > g$. □

Problem 10A,9
Suppose $b_1, b_2, \ldots$ is abounded sequence in $\mathbb{F}$. Define a bounde linear map $T : l^2 \to l^2$ by

$$T(a_1, a_2, \ldots) = (a_1 b_1, a_2 b_2, \ldots),$$

- Find a formula for T^*.
- show that T is injective if and only if $b_k \neq 0$ for every positive integer k.
- show that T has dense range if and only if $b_k \neq 0$ for every positive integer k.
- Show that T has closed tange if and only if

$$\inf\{|b_k| : k \in Z^+ and\ b_k \neq 0\} > 0.$$

- Show that T is invertible if and only if

$$\inf\{|b_k| : k \in Z^+\} > 0.$$

Proof.

- It is easy to check that $T^*(a_1, a_2, \ldots) = (\bar{b_1}a_1, \bar{b_2}a_2, \ldots)$.
- It is easy to check.
- If there exists k_0 such that $b_{k_0} = 0$, then e_{k_0} is not on the closure of range of T, where e_{k_0} only have k_0-th entry nonzero and the value is 1. The converse is direct.
- If $\inf\{|b_k| : k \in Z^+ and\ b_k \neq 0\} > 0$, without loss of generality we can assume all b_k is non-zero since we only care about image. Then with the assumption we can check range of T is the whole l^2, which is obviously closed. Conversely, if there exist a sequence positive integer k_j going to infinity such that $\lim_{j\to\infty} b_{k_j} = 0$ and $b_{k_j} > 0$, up tp a subsequence we also assume that $\sum_{j=1}^{\infty} b_{k_j}^2 < \infty$. Then consider $c_l = T(\sum_{j=1}^{l} e_{k_j})$. So c_l converge to some c in l^2. In fact, $c = (c_k), c_k = 0\ if\ k \notin \{k_j\}, c_k = b_{k_j}\ if\ k = k_j$. c is on the closure of range of T but has no preimage in l^2.
- The argument is almost the same as in the forth question.
  □

Problem 10A,10
Suppose $h \in L^\infty(\mathbb{R})$ and $M_h : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the bounded operator definied by $M_h(f) = fh$.

- Show that M_h is injective if and only if $|\{x : h(x) = 0\}| = 0$.
- Find a necessary and sufficient condition for M_h to have dense range.
- Find a necessary and sufficient condition for M_h to have closed range.
- Find a necessary and sufficient condition for M_h to be invertible

Proof. This problem is the continuous version of the previous problem so I will not provide detail proof since their proof is very similar.

- This is obvious by definition.
- The condition is $|\{x : h(x) = 0\}| = 0$. The necessary part is easy. For the sufficient part, consider the set $A_k = \{x : |h(x)| > \frac{1}{k}\}$. Then by assumption $f\chi_{A_k}$ converge in L^2 to f and the former one lies in the range of M_h.
- The condition is $\inf\{|h(x)| h(x) \neq 0\} > 0$. The sufficient part is easy. For the necessary part, we assume $\inf\{|h(x)| h(x) \neq 0\} = 0$. Let $B_k = \{x : |h(x)| \in (\frac{1}{k+1}, \frac{1}{k}]\}$. Then up to a subsequence, we assume $|B_k| > 0$ for all k. Choose $b_k > 0$ such that $\sum_{k=1}^{\infty} k^2 b_k^2 |B_k| = \infty, \sum_{k=1}^{\infty} b_k^2 |B_k| < \infty$. Then consider the sequence of function $f_j(x) = \sum_{k=1}^{j} b_k \chi_{B_k}$ and the same argument as in the previous problem gives the result.
- The condition is $\inf\{|h(x)| h(x) \neq 0\} > 0$ and $|\{x : h(x) = 0\}| = 0$. The argument is the same as before.

□